Econometrica, Vol. 35. No. 2 (April, 1967)

A MODEL OF BUSINESS FIRM GROWTH

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A new method is proposed for deriving skew distributions of business firm sizes from the assumption of Gibrat's Law. The growth of the firm is decomposed into an industry-wide component and an individual component, the latter governed by a one-period Markov process. The model is fitted to data on the recent growth of large American firms.

A NUMBER of stochastic models, embodying various forms of Gibrat's law of proportionate effect, have been shown to generate skew distribution functions resembling the actual size distributions of business firms. (See [2] and references cited there.) In a previous paper [1] we presented some results of the simulation of such a model permitting serial correlations over time in the size changes of individual firms. The aim of the present paper is to carry further the analysis of autocorrelated growth, by proposing an economically meaningful scheme for its analysis, and applying the scheme to some data on large American firms.

In studying business firm growth, we often encounter cases where a firm suddenly acquires an impetus for growth. Perhaps by innovating in production or marketing processes, or perhaps as an effect of new management staffs or techniques, the firm grows much more rapidly than the other firms in the industry, as measured, say, by the ratio of the current firm size to its size in the previous time period. Thus, we may observe that, while most of the firms in the industry are growing at, say, 5% a year, some firms grow 10%.

Furthermore, a firm that grew 10% last year is likely to grow more rapidly than average again this year as a result of the carry-over effects of an innovation that occurred in a previous year on operations in subsequent periods. This carry-over becomes more and more likely as we shorten the length of the time period we are considering from a year to a month, week, or day. Moreover, on the average, a firm which grew rapidly in one year subsequently retains a greater share of the industry assets (or market share if sales are used as a measure of firm size) from that time on than do firms that have enjoyed only the average industry growth. Therefore, not only the growth rate over and above the average growth rate, but also the period when the extra growth took place are important factors in the individual firm's growth relative to the industry growth.

In this paper, we develop a model to represent such characteristics of firms' growth, so that the process may be analysed further. In the final section we estimate the key parameter of the model for the recent growth of large American business firms.

1. THE GROWTH MODEL

Let us represent by S_{it} the size of the ith firm at the end of the tth period. The size may be measured by the total assets of the firm or by its sales volume. We shall assume that there are N firms in the industry. For convenience, we shall consider a single industry, but will show later that the analysis may be applied to the economy of a given country as a whole.

Consider the relation that defines size ratios, r_{it}:

$$S_{it}=r_{it}S_{i(t-1)} \qquad (t=1, 2, \ldots, T). \tag{1}$$

Hence, the quantity r_{it} may be called the growth ratio of the ith firm in the tth period. Let us decompose r_{it} into two factors: one, a growth factor applicable to the ith firm only (the individual growth factor), ρ_{it}, and the other, a growth factor that affects equally all firms in the industry (the industry growth factor), $\bar{\rho}_t$. Then we decompose r_{it} into ρ_{it} and $\bar{\rho}_t$ by the definitional equation:

$$r_{it}=\rho_{it}\cdot\bar{\rho}_t \qquad (t=1, 2, \ldots, T). \tag{2}$$

Hence

$$S_{it}=\rho_{it}\cdot\bar{\rho}_t S_{i(t-1)} \qquad (t=1, 2, \ldots, T). \tag{3}$$

Equations (2) and (3) are merely definitions of the growth rate factors. The industry growth ratio $\bar{\rho}_t$ affects the size of all firms in the industry equally and the individual growth factor, ρ_{it}, is the residual of the ith firm's growth that has taken place in the tth period over and above the industry growth factor.

Equation (2) defines only the product of the industry growth factor, $\bar{\rho}_t$ and the individual growth factor, ρ_{it}; this product can be decomposed into its factors in any way that seems theoretically or statistically convenient. From the standpoint of the theoretical model, the individual growth factors should be defined so as to be statistically independent of the industry growth ratio. From a practical, statistical standpoint, however, it is satisfactory to identify the industry growth ratio with the quantity $\bar{\rho}_t=\Sigma_i S_{it}/\Sigma_i S_{i(t-1)}$, that is, with the ratio of the size of the industry in the current period to its size in the previous period. Then ρ_{it} is a measure of the change in the ith firm's share of market in the industry (using sales volume to measure size); so that if $\rho_{it}=1$, the ith firm has grown just rapidly enough to retain its share of market. With this definition, the statistical dependence of the average growth ratio on any individual growth factor will be too slight to bias significantly the estimates of parameters of the model, provided, of course, that the number of firms is relatively large.

From (3) we have

$$S_{it} = \left(\prod_{\tau=1}^{t} \rho_{i\tau}\right)\left(\prod_{\tau=1}^{t} \bar{\rho}_\tau\right) S_{i0}\,. \tag{4}$$

That is,

$$\text{(5)} \qquad \log S_{it} = \sum_{\tau=1}^{t} \log \rho_{i\tau} + \sum_{\tau=1}^{t} \log \bar{\rho}_{\tau} + \log S_{i0} .$$

By means of definitions, we have attained, in equations (4) and (5), a decomposition of the size of the ith firm into a product of factors accounting for its growth. The first set of factors in the product reflects idiosyncratic events that distinguish this firm's history from the histories of other firms in the industry. The second set of factors determines the industry's growth. The final factor is the initial size of the firm.

Suppose that S_{i0} is given and $\bar{\rho}_{\tau}$, the industry growth ratio is also given for all $\tau = 1, 2, \ldots, t$. Then the only remaining factor in determining S_{it} is $\Pi \rho_{i\tau}$ ($\tau = 1, 2, \ldots, t$). We now assume that the quantities $\rho_{i\tau}$ satisfy the following hypothesis:

HYPOTHESIS: The individual growth ratio ρ_{it} of the ith firm in the tth period is the product of some power of the growth ratio $\rho_{i(t-1)}$ of the same firm in the $(t-1)$st period and a random factor ε_{it}, which is distributed independently and identically for every firm and for every t, i.e.,

$$\text{(6)} \qquad \rho_{it} = \varepsilon_{it} \rho_{i(t-1)}^{\alpha} ,$$

where α is a constant, and

$$\text{(7)} \qquad \rho_{i1} = \varepsilon_{i1} .$$

Notice that this hypothesis takes into account the following facts that we often observe in the analysis of firm growth.

(i) The expected value of the individual growth ratio is independent of the firm's size (Gibrat's Law).

(ii) The individual growth ratio in one period is related to the individual growth ratio in the previous period (a single period Markov process).

(iii) The individual growth ratio of a firm is determined independently from that of other firms. That is, factors that affect more than one firm are considered to be absorbed in the industry growth ratio $\bar{\rho}_t$.

(iv) With α in the range $0 \leqslant \alpha < 1$, an individual growth ratio in one period will have decaying effects on the ratios in subsequent periods. That is, a firm that grew more than (or less than) the industry growth rate in the previous period, namely $\rho_{i(t-1)} > 1$ (or $\rho_{i(t-1)} < 1$), on the average tends to grow more than (or less than) the industry growth rate in the current period but at a rate closer, on the average, to the industry growth rate than in the previous period, namely $|\rho_{it} - 1| < |\rho_{i(t-1)} - 1|$.

Under our hypothesis we can develop the model as below. From (6) and (7), we have

$$\text{(8)} \qquad \log \rho_{it} = \log \varepsilon_{it} + \alpha \log \rho_{i(t-1)}$$

$$= \sum_{\tau=1}^{t} \alpha^{(t-\tau)} \log \varepsilon_{i\tau} .$$

Hence

$$(9)\qquad \sum_{t=1}^{T} \log \rho_{it} = \sum_{t=1}^{T} \sum_{\tau=1}^{t} \alpha^{(t-\tau)} \log \varepsilon_{i\tau}$$

$$= (1+\alpha+\alpha^2 \ldots + \alpha^{T-1}) \log \varepsilon_{i1} + (1+\alpha+\ldots+\alpha^{T-2}) \log \varepsilon_{i2} + \ldots$$

$$+ (1+\alpha+\ldots\alpha^{T-k}) \log \varepsilon_{ik} + \ldots + \log \varepsilon_{iT}$$

$$= \sum_{t=1}^{T} \frac{1-\alpha^{T-t+1}}{1-\alpha} \log \varepsilon_{it} .$$

Thus, from (5) and (9),

$$(10)\qquad \log S_{it} = \sum_{\tau=1}^{t} \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} \log \varepsilon_{i\tau} + \sum_{\tau=1}^{t} \log \bar{\rho}_{\tau} + \log S_{i0} .$$

We may remark that in the special case where log ε_i is normally distributed with mean zero and variance σ^2, ε_i has a log normal distribution with mean 1. Let

$$(11)\qquad x_{it} = \sum_{\tau=1}^{t} \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} \log \varepsilon_{i\tau} .$$

Clearly, x_{it} is normally distributed, since it is a weighted sum of independent random variables, each of which is normally distributed. In what follows, however, we assume independence of the log ε_{it}'s, but we do not assume normality. Since the mean of log ε_i is zero, the mean of x_{it}, denoted by Ex_{it}, is also zero, for

$$(12)\qquad Ex_{it} = E\left(\sum_{\tau=1}^{t} \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} \log \varepsilon_{i\tau}\right)$$

$$= \sum_{\tau=1}^{t} \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} E\log \varepsilon_{i\tau} = 0 .$$

On the other hand, the variance of x_{it}, denoted by Dx_{it}, is given by

$$(13)\qquad Dx_{it} = D\left(\sum_{\tau=1}^{t} \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} \log \varepsilon_{i\tau}\right)$$

$$= \sum_{\tau=1}^{t} \left(\frac{1-\alpha^{(t-\tau+1)}}{1-\alpha}\right)^2 D\log \varepsilon_{i\tau}$$

$$= \frac{\sigma^2}{(1-\alpha)^2} \sum_{\tau=1}^{t} [1-2\alpha^{(t-\tau+1)} + \alpha^{2(t-\tau+1)}]$$

$$= \frac{\sigma^2}{(1-\alpha)^2} \left[t - 2\alpha \frac{1-\alpha^t}{1-\alpha} + \alpha^2 \frac{1-\alpha^{2t}}{1-\alpha^2}\right].$$

Note that

$$(14)\qquad \lim_{t\to\infty} Dx_{it} = \infty$$

and that for $0 \leqslant \alpha < 1$

$$(15) \qquad Dx_{lt} \geqslant D\left(\sum_{\tau=1}^{t} \log \varepsilon_{l\tau}\right)$$

with the equality sign holding if and only if $\alpha = 0$ or $t = 1$, since

$$(16) \qquad \frac{1-\alpha^{(t-\tau+1)}}{1-\alpha} > 1 \quad \text{for all} \quad 1 \leqslant \tau < t,\ 0 < \alpha < 1,\ t > 1.$$

Now let us return to equation (10) for $\log S_{lt}$. If the second and the third terms in the right-hand side of the equality are assumed to be determinate, the distribution function of $\log S_{lt}$ is completely determined by the distribution function of x_{lt}, except the position of the mean. Therefore, as t increases the probability density function for $\log S_{lt}$ becomes flatter and flatter, the distribution function approaching asymptotically:

$$(17) \qquad F(x) = \tfrac{1}{2}\,.$$

On the other hand, we have

$$(18) \qquad \begin{aligned} D(\log \rho_{lt}) &= D(\log \varepsilon_{lt} + \alpha \log \varepsilon_{l(t-1)} + \ldots + \alpha^{t-1} \log \varepsilon_{l1}) \\ &= D \log \varepsilon_{lt} + \alpha^2 D \log \varepsilon_{l(t-1)} + \ldots \alpha^{2t-2} D \log \varepsilon_{l1} \\ &= (1 + \alpha^2 + \ldots \alpha^{2t-2})\, \sigma^2\,. \end{aligned}$$

Thus,

$$(19) \qquad \lim_{t \to \infty} D(\log \rho_{lt}) = \frac{\sigma^2}{(1-\alpha^2)}\,.$$

2. THE MULTIPLIER

We can compare the limit of the variance of $\log \rho_{lt}$ just derived with the variance of the unweighted average, over time, of the ε's. Let us call the latter variance y_{lt}, defined by:

$$(20) \qquad y_{lt} = \frac{1}{t} \sum_{\tau=1}^{t} \log \varepsilon_{l\tau}\,.$$

Because of the independence of the ε_{lt}, we have immediately:

$$(21) \qquad Dy_{lt} = \sigma^2\,.$$

Thus α operates as a multiplier on the $\log \varepsilon_{lt}$, increasing the resulting variance in the growth ratios from σ^2 to $\sigma^2/(1-\alpha^2)$.

The empirical meaning of α may be seen in the following manner. For simplicity, assume that

$$(22) \qquad \log \bar{\rho}_\tau = c \quad \text{for all} \quad \tau = 1, 2, \ldots, t-1, t, t+1, \ldots$$

where c is a constant and

$$(23) \qquad \log \varepsilon_{l\tau} = 0 \quad \text{for all} \quad \tau = 1, 2, \ldots, t-1.$$

Then $\log S_{i\tau}$ is given by

(24) $$\log S_{i\tau} = c\tau + \log S_{i0} \quad \text{for} \quad \tau = 1, 2, \ldots, t-1.$$

Suppose that $\log \varepsilon_{it} \neq 0$, while $\varepsilon_{i\tau} = 0$ for $\tau = 1, \ldots, t-1$, and that α is equal to zero. Then the effect of $\log \rho_{it}$ is to make a parallel shift of the time path for $\log S_{i\tau}$ by the quantity $\log \varepsilon_{it}$ (see Figure 1).

When $\alpha = 0$, there is no carry-over effect on the growth in the subsequent periods, hence the line after the shift will be parallel to the original one if the subsequent terms $\log \varepsilon_{i\tau}$ for $\tau > t$, are all zero.

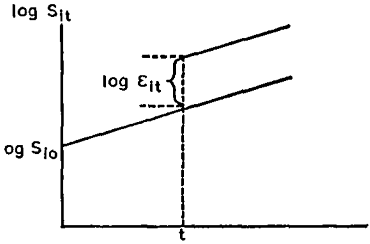

FIGURE 1

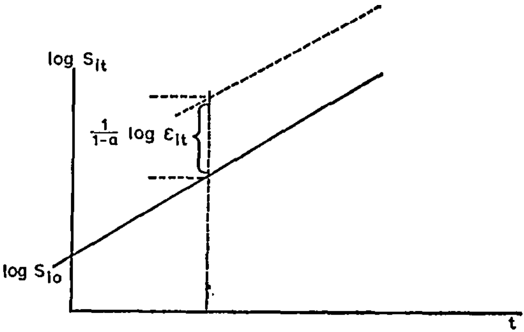

FIGURE 2

Next, consider the case where $0 < \alpha < 1$ and $\log \varepsilon_{i\tau} = 0$ for all τ except $\tau = t$. Then the time path of $\log S_{it}$ is shifted by the quantity $\log \varepsilon_{it}$ at the period t. It is shifted again in the period $t+1$ by the quantity $\alpha \log \varepsilon_{it}$; in the period $t+2$ by the quantity $\alpha^2 \log \varepsilon_{it}$, and so on. The actual growth curve, then, approaches asymptotically (Figure 2):

(25) $$\log S_{i\tau} = c\tau + \log S_{i0} + \frac{1}{1-\alpha} \log \varepsilon_{it}\,.$$

3. A TWO-LEVEL GENERALIZATION

The stochastic model is readily generalized to admit more than two causes of change in size. For example, the growth ratio for each firm might be expressed as the product of three factors: a ratio for the economy as a whole, a factor describing the growth of the firm's industry relative to the economy as a whole, and a factor expressing the growth of the individual firm relative to its industry. Then equation (2) would be replaced by:

$$(2') \qquad r_{ijt} = \rho_{ijt} \cdot \rho_{jt} \cdot \bar{\rho}_t \qquad (t = 1, 2, \ldots, T),$$

where $\bar{\rho}_t$ is the average rate for the economy, ρ_{jt} the growth factor associated with the jth industry, and ρ_{ijt} the factor associated with the ith firm in the jth industry.

Alternatively, we can combine the first two factors in (2′), arriving again at a product of two factors, the first of which reflects the joint effect of the idiosyncratic growth of the firm's industry relative to the economy and of the individual firm relative to the industry. In the next section we shall use this latter interpretation, formally identical with the original model of equation (2), to analyse some growth rates in the American economy.

4. GROWTH OF LARGE AMERICAN FIRMS

To illustrate the application of the model, we have estimated α for the recent growth of large American firms. The data are the sales of the ninety six largest American firms, obtained from the *Fortune* tabulation,[1] for the years 1954, 1958, and 1962. A four year time interval was used so that the middle-run growth trends of individual firms would not be swamped by short-run business cycle fluctuations. Defining, as above, $\bar{\rho}_t = \Sigma_i S_{it} / \Sigma_i S_{i(t-1)}$, the economy growth ratio was found to be 1.27 both for the four year period 1954–1958 and for the four year period 1958–1962. (This corresponds to a growth rate of about six per cent per annum.) These quantities, inserted in equation (3), provided estimates for the ρ_{it} for the same two time intervals (call them ρ_{i1} and ρ_{i2}, respectively). Inserting the logarithms of these growth factors in equation (8), the method of least squares was used to estimate α. The regression equation is:

$$(26) \qquad \log \rho_{i2} = .35 \log \rho_{i1} - .00034, \qquad \text{or } \alpha = .35\,.$$

Thus α, the factor measuring the degree of persistence of sudden growth, was slightly greater than one third for large American firms over a four year period. A firm that experienced an unusually rapid growth in the first four year period

[1] Data for four of the one hundred largest firms in 1962 were not usable, because the data were not available for all three years, or because large scale mergers had made the data entirely non-comparable. The smaller non-comparabilities from year to year that undoubtedly exist for some of the remaining firms were simply ignored.

could expect a greater than average growth in the second four year period. But the logarithm of the ratio measuring the excess would be, on the average, only one third as large during the second period as during the first. Thus, a firm that doubled its share of market (i.e., of the total economy) in the first four years ($\rho_{i1}=2$), could be expected, on the average, to increase its share of market by about twenty eight per cent in the second four year period (for $\log(1.28) \sim .105 = .35 (\log 2)$). Rapidly growing firms "regress" relatively rapidly to the average growth rate of the economy.

The same point may also be stated using equation (25). Since for these data, $1/(1-\alpha)=1.54$, a firm that experienced a "windfall" growth of magnitude $\log \varepsilon_{it}$ during the first four year period could expect a total effect of this "windfall" upon $\log S_{iT}$ of $1.54 \log \varepsilon_{it}$ before its growth rate returned again to the average for the economy.

Since our data provide only two time intervals for comparison, they do not allow us to test the assumption that the $\log \varepsilon_{it}$ are distributed independently for all time periods.

5. CONCLUSION

In this paper we have proposed a model of business firm growth that decomposes the growth of a firm into an industry-wide component and a component peculiar to that firm. We have developed a Markov-process model for the individual component of growth, and have shown how to estimate the key parameter of the model, a parameter measuring the persistence of spurts in growth.

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