NOTE: SOME CONDITIONS OF MACROECONOMIC STABILITY

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IN a recent paper by one of us[1] there is an error in the statement of a supposed sufficient condition that a system of linear homogeneous equations should have solutions all of the same sign. The present note is intended to correct that error, to state and prove an apparently new, necessary and sufficient condition that the stated consequence should hold, and finally to interpret the significance of this condition in economic terms.

Two preliminary remarks are in order. First, the error in the theorem originally stated does not affect the substance of the paper to which reference is being made. That paper sets forth theorems on the stability of systems which *do* have stationary solutions with all variables positive. The lemma under consideration here gives necessary and sufficient conditions that a system *will* have stationary solutions with all variables positive. That is, it gives a criterion to test whether the theorems in the body of the paper are applicable to any particular system of equation.

Second, the conditions under which the variables satisfying a system of linear equations will be all positive are of economic interest in their own right. They are, in fact, the conditions determining whether a system of linear production functions is capable, given a sufficient supply of the "fixed" factors of production, of producing any desired schedule of consumption goods.

The system of equations[2] is the following:[3]

$$\sum_{j=1}^{n} b_{ij}x_j = 0, \qquad (i = 1, \cdots, n) \tag{1}$$

with $\Delta = 0$ and of rank $n - 1$; $b_{ij} > 0$ for all $i \neq j$; $b_{ii} < 0$ for all i.

Instead of dealing directly with the system (1), it will be more convenient to consider the associated nonhomogeneous system:

$$\sum_{j=1}^{m} a_{ij}x_j - k_i = 0, \qquad (i = 1, \cdots, m) \tag{2}$$

[1] David Hawkins, "Some Conditions of Macroeconomic Stability," ECONOMETRICA, Vol. 16, October, 1948, pp. 309-322. The theorem under discussion is on page 312.

[2] The system (1) is essentially that introduced by W. W. Leontief in *The Structure of American Economy*, 1919-1929, Cambridge: Harvard University Press, 1941. p. 48.

[3] In Hawkin's original system we require only that $b_{ij} \geqslant 0$ for $i \neq j$. The stronger condition $b_{ij} > 0$ employed here simplifies the statement of the theorem and its proof and, because of the continuity of solutions of these equations with respect to variations of these coefficients, does not involve any essential loss of generality.

with

$$m = n - 1; k_i = b_{in}; a_{ij} = -b_{ij}, \quad (i, j = 1, \cdots, m); |A| = |a_{ij}| \neq 0.$$

It is clear that, for $x_n = 1$, the solution of (1) is identical with that of (2), and the $[x_i]$ satisfying (1) will all be of the same sign if and only if the $[x_i]$ satisfying (2) are all positive. Further, without loss of generality, we can take $a_{ii} = -b_{ii} = 1$.

In equations (2), x_i is the total quantity of the ith commodity produced; k_i is the quantity of the ith commodity consumed; $-a_{ij}x_j$ is the quantity of the ith commodity used in producing the jth commodity. The nth equation in system (1), which is linearly dependent upon the first m equations, may be interpreted as a consumption function. Alternatively, the vector $[k_i]$ in (2), which gives the relative quantities of the various commodities consumed, may be considered the schedule of consumption goods.

The production system (2) is economically meaningful only if the $[x_i]$ satisfying it are all positive. Conceivably, the signs of the $[x_i]$ may depend upon the magnitude of the $[k_i]$—that is, upon the schedule of consumption goods. Hence we will be interested in knowing under what conditions the $[x_i]$ will be positive for some given set $[k_i]$, and under what conditions the $[x_i]$ will be positive for *any* set $[k_i \geqslant 0]$.

The defective theorem is the following: LEMMA: *The system of equations (1) is satisfied only for x_i all of the same sign.*

This theorem is true only for $n \leqslant 3$, as shown by the following counter-example:

$$\begin{aligned}
-2x_1 + 4x_2 + x_3 + x_4 &= 0 \\
4x_1 - 2x_2 + x_3 + x_4 &= 0 \\
x_1 + x_2 - 2x_3 + 4x_4 &= 0 \\
x_1 + x_2 + 4x_3 - 2x_4 &= 0
\end{aligned}$$

We verify immediately that $\Delta = 0$, and is of rank $n - 1$, and that $b_{ij} > 0$ for all $i \neq j$, while $b_{ii} < 0$ for all i. But the general solution of this system is: $x_1 = K; x_2 = K; x_3 = -K; x_4 = -K$; where K is an arbitrary constant.

The fallacy in the proof offered for the lemma lies in paragraph III of Hawkins' paper. Specifically, it is not correct that: if all members of a set of hyperplanes intersect in a common line through the origin, and if each member of the set has points lying in the first quadrant, then the common line of intersection must lie in the first quadrant.

We now proceed to a valid, necessary and sufficient condition that the equations (2) be satisfied only for $[x_i]$ all positive. THEOREM: *A necessary and sufficient condition that the x_i satisfying* (2) *be all positive is that all principal minors of the matrix* $||a_{ij}||$ *be positive.*

To prove this theorem we first consider the augmented $m \cdot n$ matrix $||a_{ij} - k_i||$ and proceed to reduce this matrix, row by row, to triangular form. That is, by adding to each row an appropriate linear combination of the preceding rows, we obtain a matrix in which all elements to the left of the main diagonal are zero. This procedure can always be carried out step by step until a row (say the jth) is reached with a nonpositive diagonal term. It does not alter the solution of the system and does not alter the values of the principal minors consisting of the first i rows and columns ($i = 1, \cdots, m$).

Because of the arrangement of signs in our particular matrix, all elements in the first column except the first can be made zero by adding to each row an appropriate *positive* multiple of the first row. The signs of all other elements off the main diagonal will remain negative. The sign of a_{22} may remain positive or become negative. In the former case, the third and all following elements in the second column can be made zero by adding to the corresponding rows an appropriate *positive* multiple of the second row. In general, if the first i elements on the main diagonal remain positive after the first $i-1$ steps in the triangularization, then the ith step in triangularization can be carried out by adding to the remaining rows a *positive* multiple of the ith row; otherwise by adding a *negative* multiple of the ith row. We carry out the triangularization until we reach a row with a nonpositive diagonal term.

For the matrix finally obtained, we distinguish two cases: (A) all the diagonal terms in the triangular matrix are positive, (B) at least one term on the main diagonal is nonpositive (and the jth term, say, is the first nonpositive one). We now prove that in case (A) all the principal minors are positive and all the x_i are positive; while in case (B) at least one principal minor is nonpositive and at least one of the x_i is negative—a statement equivalent to our theoren..

A. In case (A) we solve the corresponding system of equations successively for $x_m, x_{m-1}, \cdots, x_1$ in terms of the k_i. Since $k_m > 0$, we must have $x_m > 0$. Since $k_{m-j} > 0$, it follows that if all $x_{m-i} > 0$ $(i < j)$, then $x_{m-j} > 0$. Hence by induction, all the x_i must be positive. But, since a triangular determinant equals the product of the elements on its main diagonal, all principal minors consisting of the first k rows and columns of the triangular matrix are positive ($k = 1, \cdots, m$). But these minors are equal to the corresponding minors of the original matrix $||a_{ij}||$.

B. In case (B), all elements to the right of the main diagonal in the jth row of the diagonalized matrix are negative, and the diagonal term is nonpositive. Suppose now that all x_i for $i > j$ are positive.

Then, since k_j is positive, x_j must be negative.[4] But the principal minor of the first j rows and columns of the triangularized matrix will be negative or zero, since the jth element in the principal diagonal is nonpositive, the others positive. Hence the corresponding minor in $||a_{ij}||$ will be nonpositive.

Since the signs of the x_i obviously do not depend on the order in which the equations are arranged before triangularization, in case (A) *all* the principal minors of $||a_{ij}||$ must be positive.

This completes the proof of the theorem. Moreover our proof gives a direct method of testing whether the x_i satisfying a given matrix are all positive.

Corollary: *A necessary and sufficient condition that the x_i satisfying* (2) *be all positive for any set $[k_i > 0]$ is that all principal minors of the matrix $||a_{ij}||$ are positive.* This corollary follows immediately from the theorem, and from the consideration that the elements of the matrix $||a_{ij}||$ are independent of the $[k_i]$.

Economic Interpretations. From the corollary, we see that if the production equations are internally consistent in permitting the production of some fixed schedule of consumption goods, then these consumption goods can be obtained in any desired proportion from this production system. Hence the system will be consistent with *any* schedule of consumption goods.

The condition that all principal minors must be positive means, in economic terms, that the group of industries corresponding to each minor must be capable of supplying more than its own needs for the group of products produced by this group of industries. If this is true, and if the condition $\Delta = 0$ for equations (1) is satisfied, then we can say that each group of industries must be just capable of supplying its own demands upon itself *and* the demands of the other industries in the economy. For example, if the principal minor involving the ith and jth commodities is negative, this means that the quantity of the ith commodity required to produce one unit of the jth commodity is greater than the quantity of the ith commodity that can be produced with an input of one unit of the jth commodity. Under these circumstances, the production of these two commodities could not be continued, for they would exhaust each other in their joint production.

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[4] Or, if the diagonal term is zero, we have a contradiction—i.e., all x for $i > j$ cannot be positive.